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# Generalizing the powerset construction, coalgebraically \*

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**ABSTRACT.** Coalgebra is an abstract framework for the uniform study of different kinds of dynamical systems. An endofunctor  $F$  determines both the type of systems ( $F$ -coalgebras) and a notion of behavioral equivalence ( $\sim_F$ ) amongst them. Many types of transition systems and their equivalences can be captured by a functor  $F$ . For example, for deterministic automata the derived equivalence is language equivalence, while for non-deterministic automata it is ordinary bisimilarity. The powerset construction is a standard method for converting a nondeterministic automaton into an equivalent deterministic one as far as language is concerned. In this paper, we lift the powerset construction on automata to the more general framework of coalgebras with structured state spaces. Examples of applications include partial Mealy machines, (structured) Moore automata, and Rabin probabilistic automata.

## 1 Introduction

*Coalgebra* is by now a well established general framework for the study of the behaviour of large classes of dynamical systems, including various kinds of automata (deterministic, probabilistic etc.) and infinite data types (streams, trees and the like). For a functor  $F: \mathbf{Set} \rightarrow \mathbf{Set}$ , an  $F$ -coalgebra is a pair  $(X, f)$ , consisting of a set  $X$  of states and a function  $f: X \rightarrow F(X)$  defining the observations and transitions of the states. Coalgebras generally come equipped with a standard notion of (behavioural) equivalence called *F-bisimilarity* that is fully determined by their (functor) type  $F$ . Moreover, for most functors  $F$  there exists a *final* coalgebra into which any  $F$ -coalgebra is mapped by a unique homomorphism that identifies all  $F$ -bisimilar states.

Much of the coalgebraic approach can be nicely illustrated with deterministic automata (DA), which are coalgebras of the functor  $D(X) = 2 \times X^A$ . In a DA, two states are  $D$ -bisimilar precisely when they accept the same language. The set  $2^{A^*}$  of all formal languages constitutes a final  $D$ -coalgebra, into which every DA is mapped by a homomorphism that sends any state to the language it accepts.

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It is well-known that *non-deterministic* automata (NDA) often provide more efficient (smaller) representations of formal languages than DA's. Language acceptance of NDA's is typically defined by turning them into DA's via the *powerset construction*. Coalgebraically this works as follows. NDA's are coalgebras of the functor  $N(X) = 2 \times \mathcal{P}_\omega(X)^A$ , where  $\mathcal{P}_\omega$  is the finite powerset. An  $N$ -coalgebra  $(X, f: X \rightarrow 2 \times \mathcal{P}_\omega(X)^A)$  is *determinized* by transforming it into a  $D$ -coalgebra  $(\mathcal{P}_\omega(X), f^\sharp: \mathcal{P}_\omega(X) \rightarrow 2 \times \mathcal{P}_\omega(X)^A)$  (for details see Section 3). Then, the language accepted by a state  $s$  in the NDA  $(X, f)$  is defined as the language accepted by the state  $\{s\}$  in the DA  $(\mathcal{P}_\omega(X), f^\sharp)$ .

For a second variation on DA's, we look at *partial automata* (PA): coalgebras of the functor  $P(X) = 2 \times (1 + X)^A$ , where for certain input letters transitions may be undefined. Again, one is often interested in the DA-behaviour (i.e., language acceptance) of PA's. This can be obtained by turning them into DA's using *totalization*. Coalgebraically, this amounts to the transformation of a  $P$ -coalgebra  $(X, f: X \rightarrow 2 \times (1 + X)^A)$  into a  $D$ -coalgebra  $(1 + X, f^\sharp: 1 + X \rightarrow 2 \times (1 + X)^A)$ .

Although the two examples above may seem very different, they are both instances of one and the same phenomenon, which it is the goal of the present paper to describe at a general level. Both with NDA's and PA's, two things happen at the same time: (i) more (or, more generally, different types of) transitions are allowed, as a consequence of changing the functor type by replacing  $X$  by  $\mathcal{P}_\omega(X)$  and  $(1 + X)$ , respectively; and (ii) the behaviour of NDA's and PA's is still given in terms of the behaviour of the original DA's (language acceptance).

For a large family of  $F$ -coalgebras, both (i) and (ii) can be captured simultaneously with the help of the categorical notion of *monad*, which generalizes the notion of algebraic theory. The structuring of the state space  $X$  can be expressed as a change of functor type from  $F(X)$  to  $F(T(X))$ . In our examples above, both the functors  $T_1(X) = \mathcal{P}_\omega(X)$  and  $T_2(X) = 1 + X$  are monads, and NDA's and PA's are obtained from DA's by changing the original functor type  $D(X)$  into  $N(X) = D(T_1(X))$  and  $P(X) = D(T_2(X))$ . Regarding (ii), one assigns  $F$ -semantics to an  $FT$ -coalgebra  $(X, f)$  by transforming it into an  $F$ -coalgebra  $(T(X), f^\sharp)$ , again using the monad  $T$ . In our examples above, the determinization of NDA's and the totalization of PA's consists of the transformation of  $N$ - and  $P$ -coalgebras  $(X, f)$  into  $D$ -coalgebras  $(T_1(X), f^\sharp)$  and  $(T_2(X), f^\sharp)$ , respectively.

We shall investigate general conditions on the functor types under which the above constructions can be applied: for one thing, one has to ensure that the  $FT$ -coalgebra map  $f$  induces a suitable  $F$ -coalgebra map  $f^\sharp$ . Our results will lead to a uniform treatment of all kinds of existing and new variations of automata, that is,  $FT$ -coalgebras, by an algebraic structuring of their state space through a monad  $T$ . Furthermore, we shall prove a number of general properties that hold in all situations similar to the ones above. For instance, there is the notion of  $N$ -bisimilarity with which NDA's, being  $N$ -coalgebras, come equipped. It coincides with the well-known notion of Park-Milner bisimilarity from process algebra. A general observation is that if two states in an NDA are  $N$ -bisimilar then they are also  $D$ - (that is, language-) equivalent. For PA's, a similar statement holds. One further contribution of this paper is a proof of these statements, once and for all for all  $FT$ -coalgebras under consideration.

Coalgebras of type  $FT$  were studied in [16, 2, 12]. In [2, 12] the main concern was def-

initions by coinduction, whereas in [16] a proof principle was also presented. All in all, the present paper can be seen as the understanding of the aforementioned papers from a new perspective, presenting a uniform view on various automata constructions and equivalences.

The structure of the paper is as follows. After preliminaries (Section 2) and the details of the motivating examples above (Section 3), Section 4 presents the general construction as well as many more examples. In Section 5, a large family of automata (technically: functors) is characterized to which the constructions above can be applied. Section 6 discusses related work and presents pointers to future work. In Appendix A, we further prove the expressivity of our framework by showing that it can subsume many behavioural equivalences from the so called *linear-time branching-time spectrum* [7].

## 2 Background

In this section we introduce the preliminaries on coalgebras and algebras. First, we fix some notation on sets. We will denote sets by capital letters  $X, Y, \dots$  and functions by lower case letters  $f, g, \dots$ . Given sets  $X$  and  $Y$ ,  $X \times Y$  is the cartesian product of  $X$  and  $Y$  (with the usual projection maps  $\pi_1$  and  $\pi_2$ ),  $X + Y$  is the disjoint union (with injection maps  $\kappa_1$  and  $\kappa_2$ ) and  $X^Y$  is the set of functions  $f: Y \rightarrow X$ . The collection of finite subsets of  $X$  is denoted by  $\mathcal{P}_\omega(X)$ , while the collection of full-probability distributions with finite support is  $\mathcal{D}_\omega(X) = \{f: X \rightarrow [0, 1] \mid f \text{ finite support and } \sum_{x \in X} f(x) = 1\}$ . For a set of letters  $A$ ,  $A^*$  denotes the set of all words over  $A$ ;  $\epsilon$  the empty word; and  $w_1 \cdot w_2$  (and  $w_1 w_2$ ) the concatenation of words  $w_1, w_2 \in A^*$ .

**Coalgebras** A coalgebra is a pair  $(X, f: X \rightarrow F(X))$ , where  $X$  is a set of states and  $F: \mathbf{Set} \rightarrow \mathbf{Set}$  is a functor. The functor  $F$ , together with the function  $f$ , determines the *transition structure* (or dynamics) of the  $F$ -coalgebra [23].

An  $F$ -homomorphism from an  $F$ -coalgebra  $(X, f)$  to an  $F$ -coalgebra  $(Y, g)$  is a function  $h: X \rightarrow Y$  preserving the transition structure, i.e.,  $g \circ h = F(h) \circ f$ .

An  $F$ -coalgebra  $(\Omega, \omega)$  is said to be *final* if for any  $F$ -coalgebra  $(X, f)$  there exists a unique  $F$ -homomorphism  $\llbracket - \rrbracket_X: X \rightarrow \Omega$ . All the functors considered in examples in this paper have a final coalgebra.

Let  $(X, f)$  and  $(Y, g)$  be two  $F$ -coalgebras. We say that the states  $x \in X$  and  $y \in Y$  are *bisimilar*, written  $x \sim_F y$ , if and only if they are mapped into the same element in the final coalgebra, that is  $\llbracket x \rrbracket_X = \llbracket y \rrbracket_Y^\dagger$ .

**Algebras** Monads can be thought of as a generalization of algebraic theories. A *monad*  $\mathbf{T} = (T, \mu, \eta)$  is a triple consisting of an endofunctor  $T$  on  $\mathbf{Set}$  and two natural transformations: a *unit*  $\eta: Id \Rightarrow T$  mapping a set  $X$  to its free algebra  $T(X)$ , and a *multiplication*  $\mu: T^2 \Rightarrow T$ . They satisfy the following commutative laws

$$\mu \circ \eta_T = id_T = \mu \circ T\eta \quad \text{and} \quad \mu \circ \mu_T = \mu \circ T\mu.$$

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<sup>†</sup>Bisimilarity is usually defined in the literature in a slight different way. The definition we present here is often called *behavioural equivalence*. For most functors both notions coincide and we choose the notion of equivalence which is more convenient for presenting our story.

Sometimes it is more convenient to represent a monad  $\mathbf{T}$ , equivalently, as a *Kleisli triple*  $(T, (-)^\sharp, \eta)$  [18], where  $T$  assigns a set  $T(X)$  to each set  $X$ , the unit  $\eta$  assigns a function  $\eta_X: X \rightarrow T(X)$  to each set  $X$ , and the extension operation  $(-)^\sharp$  assigns to each  $f: X \rightarrow T(Y)$  a function  $f^\sharp: T(X) \rightarrow T(Y)$ , such that,

$$f^\sharp \circ \eta_X = f \quad (\eta_X)^\sharp = id_{T(X)} \quad (g^\sharp \circ f)^\sharp = g^\sharp \circ f^\sharp,$$

for  $g: Y \rightarrow T(Z)$ . Monads are frequently referred to as *computational types* [19]. We list now a few examples. In what follows,  $f: X \rightarrow T(Y)$  and  $c \in T(X)$ .

**Nondeterminism**  $T(X) = \mathcal{P}_\omega(X)$ ;  $\eta_X$  is the singleton map  $x \mapsto \{x\}$ ;  $f^\sharp(c) = \bigcup_{x \in c} f(x)$ .

**Partiality**  $T(X) = 1 + X$  where  $1 = \{*\}$  represents a terminating (or diverging) computation;  $\eta_X$  is the injection map  $\kappa_2: X \rightarrow 1 + X$ ;  $f^\sharp(\kappa_1(*)) = \kappa_1(*)$  and  $f^\sharp(\kappa_2(x)) = f(x)$ .

Further examples of monads include: exceptions ( $T(X) = E + X$ ), side-effects ( $T(X) = (S \times X)^S$ ), interactive output ( $T(X) = \mu v.X + (O \times v) \cong O^* \times X$ ) and full-probability ( $T(X) = \mathcal{D}_\omega(X)$ ). We will use all these monads in our examples and we will define  $\eta_X$  and  $f^\sharp$  for each later in Section 4.1.

A  **$\mathbf{T}$ -algebra** of a monad  $\mathbf{T}$  is a pair  $(X, h)$  consisting of a set  $X$ , called carrier, and a function  $h: T(X) \rightarrow X$  such that  $h \circ \mu_X = h \circ Th$  and  $h \circ \eta_X = id_X$ . A  $T$ -homomorphism between two  $\mathbf{T}$ -algebras  $(X, h)$  and  $(Y, k)$  is a function  $f: X \rightarrow Y$  such that  $f \circ h = k \circ Tf$ .  $\mathbf{T}$ -algebras and their homomorphisms form the so-called *Eilenberg-Moore category*  $\mathbf{Set}^{\mathbf{T}}$ . There is a forgetful functor  $U^{\mathbf{T}}: \mathbf{Set}^{\mathbf{T}} \rightarrow \mathbf{Set}$  defined by

$$U^{\mathbf{T}}((X, h)) = X \quad \text{and} \quad U^{\mathbf{T}}(f: (X, h) \rightarrow (Y, k)) = f: X \rightarrow Y.$$

The forgetful functor  $U^{\mathbf{T}}$  has left adjoint  $X \mapsto (T(X), \mu_X: TT(X) \rightarrow T(X))$ , mapping a set  $X$  to its free  $\mathbf{T}$ -algebra. If  $f: X \rightarrow Y$  with  $(Y, h)$  a  $\mathbf{T}$ -algebra, the unique  $\mathbf{T}$ -homomorphism  $f^\sharp: (T(X), \mu_X) \rightarrow (Y, h)$  with  $f^\sharp \circ \eta_X = f$  is given by

$$f^\sharp: T(X) \xrightarrow{Tf} T(Y) \xrightarrow{h} Y.$$

The function  $f^\sharp: (T(X), \mu_X) \rightarrow (T(Y), \mu_Y)$  coincides with function extension for a Kleisli triple. For the monad  $\mathcal{P}_\omega$  the associated Eilenberg-Moore category is the category of join semi-lattices, whereas for the monad  $1 + -$  is the category of pointed sets.

### 3 Motivating examples

In this section, we introduce two motivating examples. We will present two constructions, the determinization of a non-deterministic automaton and the totalization of a partial automaton, which we will later show to be an instance of the same, more general, construction.

#### 3.1 Non-deterministic automata

A deterministic automaton (DA) over the input alphabet  $A$  is a pair  $(X, \langle o, t \rangle)$ , where  $X$  is a set of states and  $\langle o, t \rangle: X \rightarrow 2 \times X^A$  is a function with two components:  $o$ , the output function, determines if a state  $x$  is final ( $o(x) = 1$ ) or not ( $o(x) = 0$ ); and  $t$ , the transition function, returns for each input letter  $a$  the next state. DA's are coalgebras for the functor

$2 \times Id^A$ . The final coalgebra of this functor is  $(2^{A^*}, \langle \epsilon, (-)_a \rangle)$  where  $2^{A^*}$  is the set of languages over  $A$  and  $\langle \epsilon, (-)_a \rangle$ , given a language  $L$ , determines whether or not the empty word is in the language ( $\epsilon(L) = 1$  or  $\epsilon(L) = 0$ , resp.) and, for each input letter  $a$ , returns the *derivative* of  $L$ :  $L_a = \{w \in 2^{A^*} \mid aw \in L\}$ . From any DA, there is a unique map  $l$  into  $2^{A^*}$  which assigns to each state its behaviour (that is, the language that the state recognizes).

$$\begin{array}{ccc} X & \xrightarrow{\quad l \quad} & 2^{A^*} \\ \langle o, t \rangle \downarrow & & \downarrow \langle \epsilon, (-)_a \rangle \\ 2 \times X^A & \xrightarrow{id \times l^A} & 2 \times (2^{A^*})^A \end{array}$$

A non-deterministic automaton (NDA) is similar to a DA but the transition function gives a set of next-states for each input letter instead of a single state. Thus, an NDA over the input alphabet  $A$  is a pair  $(X, \langle o, \delta \rangle)$ , where  $X$  is a set of states and  $\langle o, \delta \rangle: X \rightarrow 2 \times (\mathcal{P}_\omega(X))^A$  is a pair of functions with  $o$  as before and where  $\delta$  determines for each input letter  $a$  a set of possible next states. In order to compute the language recognized by a state  $x$  of an NDA  $\mathcal{A}$ , it is usual to first determinize it, constructing a DA  $\mathbf{det}(\mathcal{A})$  where the state space is  $\mathcal{P}_\omega(X)$ , and then compute the language recognized by the state  $\{x\}$  of  $\mathbf{det}(\mathcal{A})$ . Next, we describe in coalgebraic terms how to construct the automaton  $\mathbf{det}(\mathcal{A})$ .

Given an NDA  $\mathcal{A} = (X, \langle o, \delta \rangle)$ , we construct  $\mathbf{det}(\mathcal{A}) = (\mathcal{P}_\omega(X), \langle \bar{o}, t \rangle)$ , where, for all  $Y \in \mathcal{P}_\omega(X)$ ,  $a \in A$ , the functions  $\bar{o}: \mathcal{P}_\omega(X) \rightarrow 2$  and  $t: \mathcal{P}_\omega(X) \rightarrow \mathcal{P}_\omega(X)^A$  are

$$\bar{o}(Y) = \begin{cases} 1 & \exists y \in Y o(y) = 1 \\ 0 & \text{otherwise} \end{cases} \quad t(Y)(a) = \bigcup_{y \in Y} \delta(y)(a).$$

The automaton  $\mathbf{det}(\mathcal{A})$  is such that the language  $l(\{x\})$  recognized by  $\{x\}$  is the same as the one recognized by  $x$  in the original NDA  $\mathcal{A}$  (more generally, the language recognized by state  $X$  of  $\mathbf{det}(\mathcal{A})$  is the union of the languages recognized by each state  $x$  of  $\mathcal{A}$ ).

We summarize the situation above with the following commuting diagram:

$$\begin{array}{ccc} X & \xrightarrow{\{\cdot\}} & \mathcal{P}_\omega(X) \xrightarrow{\quad l \quad} 2^{A^*} \\ \langle o, \delta \rangle \downarrow & \swarrow \langle \bar{o}, t \rangle & \downarrow \langle \epsilon, (-)_a \rangle \\ 2 \times \mathcal{P}_\omega(X)^A & \xrightarrow{id \times l^A} & 2 \times (2^{A^*})^A \end{array}$$

We note that the language semantics of NDA's, presented in the above diagram, can also be obtained as an instance of the abstract definition scheme of  $\lambda$ -coinduction [2, 12].

### 3.2 Partial automata

A partial automaton (PA) over the input alphabet  $A$  is a pair  $(X, \langle o, \partial \rangle)$  consisting of a set of states  $X$  and a pair of functions  $\langle o, \partial \rangle: X \rightarrow 2 \times (1 + X)^A$ , with  $o: X \rightarrow 2$  as for DA and  $\partial: X \rightarrow (1 + X)^A$  a transition function, which for any input letter  $a$  is either undefined (no  $a$ -labelled transition takes place) or specifies the next state that is reached. PA's are coalgebras for the functor  $2 \times (1 + Id)^A$ . Given a PA  $\mathcal{A}$ , we can construct a total (deterministic) automaton  $\mathbf{tot}(\mathcal{A})$  by adding an extra *sink* state to the state space: every undefined  $a$ -transition from



a state  $x$  is then replaced by a  $a$ -labelled transition from  $x$  to the sink state. More precisely, given a PA  $\mathcal{A} = (X, \langle o, \partial \rangle)$ , we construct  $\mathbf{tot}(\mathcal{A}) = (1 + X, \langle \bar{o}, t \rangle)$ , where

$$\begin{aligned} \bar{o}(\kappa_1(*)) &= 0 & t(\kappa_1(*))(a) &= \kappa_1(*) \\ \bar{o}(\kappa_2(x)) &= o(x) & t(\kappa_2(x))(a) &= \partial(x)(a) \end{aligned}$$

The language  $l(x)$  recognized by a state  $x$  will be precisely the language recognized by  $x$  in the original partial automaton. Moreover, the new sink state recognizes the empty language. Again we summarize the situation above with the help of following commuting diagram, which illustrates the similarities between both constructions:

$$\begin{array}{ccccc} X & \xrightarrow{\kappa_2} & 1 + X & \xrightarrow{l} & 2^{A^*} \\ \langle o, \partial \rangle \downarrow & \swarrow \langle \bar{o}, t \rangle & & & \downarrow \langle \epsilon, (-)_a \rangle \\ 2 \times (1 + X)^A & \xrightarrow{id \times l^A} & 2 \times (2^{A^*})^A & & \end{array}$$

## 4 Algebraically structured coalgebras

In this section we present a general framework where both motivating examples can be embedded and uniformly studied. We will consider coalgebras for which the functor type  $FT$  can be decomposed into a transition type  $F$  specifying the relevant dynamics of a system and a monad  $T$  providing the state space with an algebraic structure. For simplicity, we fix our base category to be **Set**, but all results below can be generalized to an arbitrary category **C** with enough limits.

We will study coalgebras  $f: X \rightarrow FT(X)$  for a functor  $F$  and a monad  $\mathbf{T}$  such that  $FT(X)$  is a  $\mathbf{T}$ -algebra, that is  $FT(X)$  is the carrier of a  $\mathbf{T}$ -algebra  $(FT(X), h)$ . In the motivating examples,  $F$  would be instantiated to  $2 \times Id^A$  (in both) and  $T$  to  $\mathcal{P}_\omega$ , for NDAs, and to  $1 + -$  for PAs. The condition that  $FT(X)$  is a  $\mathbf{T}$ -algebra would amount to require that  $2 \times \mathcal{P}_\omega(X)^A$  is a join-semilattice, for NDAs, and that  $2 \times (1 + X)^A$  is a pointed set, for PAs. This is indeed the case, since the set  $2$  can be regarded both as a join-semilattice ( $2 \cong \mathcal{P}_\omega(1)$ ) or as a pointed set ( $2 \cong 1 + 1$ ) and, moreover, products and exponentials preserve the algebra structure.

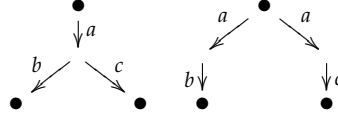
The inter-play between the transition type  $F$  and the computational type  $\mathbf{T}$  (more precisely, the fact that  $FT(X)$  is a  $\mathbf{T}$ -algebra) will allow each coalgebra  $f: X \rightarrow FT(X)$  to be extended uniquely to a  $T$ -algebra morphism  $f^\sharp: (T(X), \mu_X) \rightarrow (FT(X), h)$  which makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ f \downarrow & \swarrow f^\sharp & \\ FT(X) & & \end{array} \quad f^\sharp \circ \eta_X = f$$

Intuitively,  $\eta_X: X \rightarrow T(X)$  is the inclusion of the state space of the coalgebra  $f: X \rightarrow FT(X)$  into the structured state space  $T(X)$ , and  $f^\sharp: T(X) \rightarrow FT(X)$  is the extension of the coalgebra  $f$  to  $T(X)$ .

Next, we will study the behaviour of a given state or, more generally, we would like to say when two states  $x_1$  and  $x_2$  are equivalent. The obvious choice for an equivalence would be bisimilarity, which arises from the functor determining the type of the coalgebra –  $FT$ .

However, this equivalence is not exactly what we are looking for. In the motivating example of non-deterministic automata we wanted two states to be equivalent if they recognize the same language. If we would take the equivalence arising from the functor  $2 \times \mathcal{P}_\omega(\text{Id})$  we would be distinguishing states that recognize the same language but have difference branching types, as in the following example.



We now define a new equivalence, which will *absorb* the effect of the monad  $T$ .

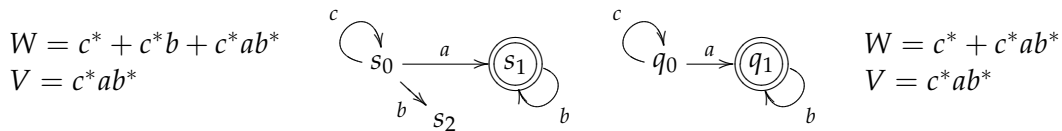
We say that two elements  $x_1$  and  $x_2$  in  $X$  are *F-equivalent with respect to a monad  $T$* , written  $x_1 \approx_F^T x_2$ , if and only if  $\eta_X(x_1) \sim_F \eta_X(x_2)$ . The equivalence  $\sim_F$  is just bisimilarity for the  $F$ -coalgebra  $f^\sharp: T(X) \rightarrow FT(X)$ .

If the functor  $F$  has a final coalgebra  $(\Omega, \omega)$ , we can capture the semantic equivalence above in the following commuting diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & T(X) & \xrightarrow{[-]} & \Omega \\ f \downarrow & & \swarrow f^\sharp & & \downarrow \omega \\ FT(X) & \xrightarrow{F[-]} & F(T(X)) & \xrightarrow{F[-]} & F(\Omega) \end{array} \quad (1)$$

Back to our first example, two states  $x_1$  and  $x_2$  of a NDA (in which  $T$  is instantiated to  $\mathcal{P}_\omega$  and  $F$  to  $2 \times \text{Id}^A$ ) would satisfy  $x_1 \approx_F^T x_2$  if and only if they recognize the same language (recall that the final coalgebra of the functor  $2 \times \text{Id}^A$  is  $2^{A^*}$ ).

It is also interesting to remark the difference between the two equivalences in the case of partial automata. The coalgebraic semantics of PAs [25] is given in terms of pairs of prefix-closed languages  $\langle V, W \rangle$  where  $V$  contains the words that are accepted (that is, are the label of a path leading to a final state) and  $W$  contains all words that label any path (that is all that are in  $V$  plus the words labeling paths leading to non-final states). We exemplify what  $V$  and  $W$  would be in the following examples for state  $s_0$  and  $q_0$ .



Thus, state  $s_0$  and  $q_0$  would be distinguished by  $FT$ -bisimilarity (for  $F = 2 \times \text{Id}^A$  and  $T = 1 + -$ ) but they are equivalent with respect to the monad  $1 + -$ ,  $s_0 \approx_F^T q_0$ , since they accept the same language.

We will show in Section 5 that the equivalence  $\sim_{FT}$  is contained in  $\approx_F^T$ .

## 4.1 Examples

In this section we show more examples of applications of the framework above.

**Partial Mealy machines** A partial Mealy machine is a set of states  $X$  together with a function  $t: X \rightarrow (B \times (1 + X))^A$ , where  $A$  is a set of inputs and  $B$  is a set of output values (with

a distinguished value  $\perp$ ). For each state  $s$  and for each input  $a$  the automaton produces an output value and either terminates or continues to a next state. Applying the framework above we will be *totalizing* the automaton, similarly to what happened in the example of partial automata, by adding an extra state to the state space which will act as a sink state. The behaviour of the totalized automaton is given by the set of causal functions from  $A^\omega$  (infinite sequences of  $A$ ) to  $B^\omega$ , which we denote by  $\Gamma(A^\omega, B^\omega)$  [24]. A function  $f: A^\omega \rightarrow B^\omega$  is causal if, for  $\sigma \in A^\omega$ , the  $n$ -th value of the output stream  $f(\sigma)$  depends only on the first  $n$  values of the input stream  $\sigma$ .

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & 1 + X \\
 \downarrow t & \swarrow t^\sharp & \\
 (B \times (1 + X))^A & \xrightarrow{\quad} & \Gamma(A^\omega, B^\omega)
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \llbracket - \rrbracket \text{---} \\
 \begin{array}{l}
 \llbracket \kappa_1(*) \rrbracket(\sigma) = (\perp, \perp, \dots) \\
 \llbracket \kappa_2(s) \rrbracket(a:\sigma) = b: (\llbracket n \rrbracket(\sigma)) \\
 \text{where } \langle b, n \rangle = t(s)(a)
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \downarrow \\
 (B \times \Gamma(A^\omega, B^\omega))^A
 \end{array}$$

**Structured Moore automata** In the following examples we look at the functor  $F(X) = T(B) \times X^A$ , for  $B$  and  $A$  arbitrary sets and  $\mathbf{T} = (T, \eta, (-)^\sharp)$  an arbitrary monad. This represents Moore automata with outputs in  $T(B)$  and inputs in  $A$ . For any set  $X$ ,  $FT(X)$  has a  $\mathbf{T}$ -algebra lifting and the final coalgebra of  $F$  is  $T(B)^{A^*}$ . The final map  $\llbracket - \rrbracket: T(X) \rightarrow T(B)^{A^*}$  is defined below.

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & T(X) \\
 \downarrow \langle o, t \rangle & \swarrow \langle o, t \rangle^\sharp & \\
 T(B) \times (T(X))^A & \xrightarrow{\quad} & T(B)^{A^*}
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \llbracket - \rrbracket \text{---} \\
 \begin{array}{l}
 \llbracket m \rrbracket(\epsilon) = o^\sharp(m) \\
 \llbracket m \rrbracket(aw) = \llbracket t^\sharp(m)(a) \rrbracket(w)
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \downarrow \langle \epsilon, (-)_a \rangle \\
 T(B) \times (T(T(B)^{A^*}))^A
 \end{array}$$

**Moore automata with exceptions.** Consider  $T(X) = E + X$ , with  $E$  a set of exceptions,  $\eta(x) = \kappa_2(x)$  and, for a function  $f: X \rightarrow T(Y)$ ,  $f^\sharp: T(X) \rightarrow T(Y)$  is defined as  $f^\sharp = [id, f]$ .

An  $FT$ -coalgebra  $\langle o, t \rangle: X \rightarrow (E + B) \times (E + X)^A$  will associate with every state  $s$  an output value (either in  $B$  or an exception in  $E$ ) and, for each input  $a$ , a next state or an exception. The behaviour of a state  $x$ , given by  $\llbracket \eta(x) \rrbracket$ , will be a formal power series over  $A$  with output values in  $E + B$  (that is, a function from  $A^*$  to  $E + B$ ), defined as follows.

$$\llbracket \kappa_1(e) \rrbracket(w) = \kappa_1(e) \quad \llbracket \kappa_2(s) \rrbracket(\epsilon) = o(s) \quad \llbracket \kappa_2(s) \rrbracket(aw) = \llbracket t(s)(a) \rrbracket(w)$$

**Moore automata with side effects.** Consider  $T(X) = (S \times X)^S$ , with  $S$  a set of side-effects,  $\eta(x) = \lambda s. \langle s, x \rangle$  and, for a function  $f: X \rightarrow T(Y)$ ,  $f^\sharp: T(X) \rightarrow T(Y)$  is defined as  $f^\sharp(g)(s) = f(x)(s')$  where  $\langle s', x \rangle = g(s)$ .

Take now an  $FT$ -coalgebra  $\langle o, t \rangle: X \rightarrow (B \times S)^S \times ((S \times X)^S)^A$  and let us explain the intuition behind this automaton type. Let  $S$  be the set of side effects (for instance, one could take  $S = V^L$ , functions associating memory locations to values). The set  $S \times X$  can be interpreted as the configurations of the automaton, where  $S$  contains information about the state of the system and  $X$  about the control of the system. Then, we can think of  $o: X \rightarrow (S \times B)^S$  as a function that for each configuration  $S \times X$  provides an output and the new state of the system (note that  $X \rightarrow (S \times B)^S \cong S \times X \rightarrow S \times B$ ). The transition function  $t: X \rightarrow ((S \times X)^S)^A$  gives a new configuration for each input letter and current configuration (again we use the fact that  $X \rightarrow ((S \times X)^S)^A \cong S \times X \rightarrow (S \times X)^A$ ).

The behaviour of a state  $x$  will be given by  $\llbracket \eta(x) \rrbracket$ , defined below, and it will be a function that for each configuration and for each sequence of actions returns an output value and a side effect.

$$\begin{aligned}\llbracket g \rrbracket(\epsilon)(s) &= o(x)(s') \text{ where } \langle s', x \rangle = g(s) \\ \llbracket g \rrbracket(aw_1) &= \llbracket \lambda s. t(s)(a)(s') \rrbracket(w_1) \text{ where } \langle s', x \rangle = g(s)\end{aligned}$$

**Moore automata with interactive output.** Consider  $T(X) = \mu v. X + (O \times v) \cong O^* \times X$ , with  $O$  a set of outputs,  $\eta(x) = \langle \epsilon, x \rangle$  and, for  $f: X \rightarrow T(Y)$ ,  $f^\sharp: T(X) \rightarrow T(Y)$  is given by  $f^\sharp(\langle w, x \rangle) = \langle ww', x' \rangle$  where  $\langle w', x' \rangle = f(x)$ . Take an *FT*-coalgebra  $\langle o, t \rangle: X \rightarrow (O^* \times B) \times (O^* \times X)^A$ . For  $B = 1$ , this coincides with a (total) *subsequential transducer* [9]:  $o: X \rightarrow O^*$  is the terminal output function;  $t: X \rightarrow (O^* \times X)^A$  is the pairing of the output function and the next state-function.

The behaviour of a state  $x$  will be given by  $\llbracket \eta(x) \rrbracket = \llbracket \langle \epsilon, x \rangle \rrbracket$ , where, for every  $\langle w, x \rangle \in O^* \times X$ ,  $\llbracket \langle w, x \rangle \rrbracket: A^* \rightarrow B^*$ , is given by

$$\llbracket \langle w, x \rangle \rrbracket(\epsilon) = w \cdot o(x) \quad \llbracket \langle w, x \rangle \rrbracket(aw_1) = w \cdot (\llbracket t(x)(a) \rrbracket(w_1))$$

**Probabilistic Moore automata.** Take  $T(X) = \mathcal{D}_\omega(X)$ ,  $\eta$  the Dirac distribution (defined below) and, for  $f: X \rightarrow T(Y)$ ,  $f^\sharp: T(X) \rightarrow T(Y)$  is given by

$$f^\sharp(c) = \lambda y. \sum_{d \in \mathcal{D}_\omega(Y)} \left( \sum_{x \in f^{-1}(d)} c(x) \right) \times d(y) \quad \eta(x) = \lambda x'. \begin{cases} 1 & x = x' \\ 0 & \text{otherwise} \end{cases}$$

Take an *FT*-coalgebra  $\langle o, t \rangle: X \rightarrow \mathcal{D}_\omega(B) \times \mathcal{D}_\omega(X)^A$ . For  $B = 2$  (note that  $\mathcal{D}_\omega(2) \cong [0, 1]$ ) this gives rise to a (Rabin) *probabilistic automaton* [22]: each state  $x$  has an output value in  $o(x) \in [0, 1]$  and, for each input  $a$ ,  $t(x)(a)$  is a probability distribution of next states. The behaviour of a state  $x$  is given by  $\llbracket \eta(x) \rrbracket: A^* \rightarrow [0, 1]$ , defined below. Intuitively, one can think of  $\llbracket \eta(x) \rrbracket$  as a probabilistic language: each word is associated with a value  $p \in [0, 1]$ .

$$\begin{aligned}\llbracket d \rrbracket(\epsilon) &= \sum_{b \in [0, 1]} \left( \sum_{o(x)=b} d(x) \right) \times b \\ \llbracket d \rrbracket(aw) &= \llbracket \lambda x'. \sum_{c \in \mathcal{D}_\omega(X)} (\sum_{b=t(x)(a)} d(x)) \times c(x') \rrbracket(w)\end{aligned}$$

It is worth to note that this exactly captures the semantics of [22], while the ordinary  $\sim_{FT}$  coincides with *probabilistic bisimilarity* of [15].

## 5 Coalgebras and T-Algebras

In the previous section we presented a framework, parameterized by a functor  $F$  and a monad  $\mathbf{T}$ , in which systems of type *FT* (that is, *FT*-coalgebras) can be studied using a novel equivalence  $\approx_F^T$  instead of the classical bisimilarity  $\sim_{FT}$ . The only requirement we imposed was that  $FT(X)$  has to be a  $\mathbf{T}$ -algebra.

In this section, we will present functors  $F$  for which the requirement of  $FT(X)$  being a  $\mathbf{T}$ -algebra is guaranteed because they can be *lifted* to a functor  $F^*$  on  $\mathbf{T}$ -algebra. For these functors, the equivalence  $\approx_F^T$  coincides with  $\sim_{F^*}$ . In other words, working on *FT*-coalgebras in **Set** under the novel  $\approx_F^T$  equivalence is the same as working on  $F^*$ -coalgebras on  $\mathbf{T}$ -algebras

under the ordinary  $\sim_{F^*}$  equivalence. Next, we will prove that for this class of functors and an arbitrary monad  $\mathbf{T}$  the equivalence  $\sim_{FT}$  is contained in  $\approx_F^T$ . Instantiating this result for our first motivating example of non-deterministic automata will yield the well known fact that bisimilarity implies trace equivalence.

Let  $\mathbf{T}$  be a monad. An endofunctor  $F^*: \mathbf{Set}^{\mathbf{T}} \rightarrow \mathbf{Set}^{\mathbf{T}}$  is said to be the  *$\mathbf{T}$ -algebra lifting* of a functor  $F: \mathbf{Set} \rightarrow \mathbf{Set}$  if the following square commutes<sup>‡</sup>:

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{T}} & \xrightarrow{F^*} & \mathbf{Set}^{\mathbf{T}} \\ U^{\mathbf{T}} \downarrow & & \downarrow U^{\mathbf{T}} \\ \mathbf{Set} & \xrightarrow{F} & \mathbf{Set} \end{array}$$

If the functor  $F$  has a  $\mathbf{T}$ -algebra lifting  $F^*$  then  $FT(X)$  is the carrier of the algebra  $F^*(T(X), \mu)$ . Functors that have a  $\mathbf{T}$ -algebra lifting are given, for example, by those endofunctors on  $\mathbf{Set}$  constructed inductively by the following grammar

$$F::= Id \mid B \mid F \times F \mid F^A \mid TG$$

where  $A$  is an arbitrary set,  $B$  is the constant functor mapping every set  $X$  to the carrier of a  $\mathbf{T}$ -algebra  $(B, h)$ , and  $G$  is an arbitrary functor. Since the forgetful functor  $U^{\mathbf{T}}: \mathbf{Set}^{\mathbf{T}} \rightarrow \mathbf{Set}$  creates and preserves limits, both  $F_1 \times F_2$  and  $F^A$  have a  $\mathbf{T}$ -algebra lifting if  $F$ ,  $F_1$ , and  $F_2$  have. Finally,  $TG$  has a  $\mathbf{T}$ -algebra lifting for every endofunctor  $G$  given by the assignment  $(X, h) \mapsto (TGX, \mu_{TG})$ . Note that we do not allow taking coproducts in the above grammar, because coproducts of  $\mathbf{T}$ -algebras are not preserved in general by the forgetful functor  $U^{\mathbf{T}}$ . Instead, one could resort to extending the grammar with the carrier of the coproduct taken directly in  $\mathbf{Set}^{\mathbf{T}}$ . For instance, if  $\mathbf{T}$  is the (finite) powerset monad, then we could extend the above grammar with the functor  $F_1 \oplus F_2 = F_1 + F_2 + \{\top, \perp\}$ .

Now, let  $F$  be a functor with a  $\mathbf{T}$ -algebra lifting and for which a final coalgebra  $\Omega$  exists. If  $\Omega$  can be constructed as the limit of the final sequence (for example assuming the functor accessible [1]), then, because the forgetful functor  $U^{\mathbf{T}}: \mathbf{Set}^{\mathbf{T}} \rightarrow \mathbf{Set}$  preserves and creates limits,  $\Omega$  is the carrier of a  $\mathbf{T}$ -algebra, and it is the final coalgebra of the lifted functor  $F^*$ . Further, for any  $FT$ -coalgebra  $f: X \rightarrow FT(X)$ , the unique  $F$ -coalgebra homomorphism  $\llbracket - \rrbracket$  as in diagram (1) is a  $T$ -algebra homomorphism between  $T(X)$  and  $\Omega$ . Conversely, the carrier of the final  $F^*$ -coalgebra (in  $\mathbf{Set}^{\mathbf{T}}$ ) is the final  $F$ -coalgebra (in  $\mathbf{Set}$ ).

Intuitively, the above means that for an accessible functor  $F$  with a  $\mathbf{T}$ -algebra lifting  $F^*$ ,  $F^*$ -equivalence in  $\mathbf{Set}^{\mathbf{T}}$  coincides with  $F$ -equivalence with respect to  $\mathbf{T}$  in  $\mathbf{Set}$ . The latter equivalence is coarser than the  $FT$ -equivalence in  $\mathbf{Set}$ , as stated in the following theorem.

**THEOREM 1.** *Let  $\mathbf{T}$  be a monad. If  $F$  is an endofunctor on  $\mathbf{Set}$  with a  $\mathbf{T}$ -algebra lifting, then  $\sim_{FT}$  implies  $\approx_F^T$ .*

The proof of this theorem (presented in appendix) relies on the fact that for every monad  $\mathbf{T}$  and functor  $F$  with a  $\mathbf{T}$ -algebra lifting, if  $h: (X, f) \rightarrow (Y, g)$  is an  $FT$ -coalgebra homomorphism, then  $(\eta_Y \circ h)^{\sharp}: (T(X), f^{\sharp}) \rightarrow (T(Y), g^{\sharp})$  is an  $F$ -coalgebra homomorphism.

The above theorem instantiates to the well-known facts: for NDA, where  $F(X) = 2 \times X^A$  and  $T = \mathcal{P}_{\omega}$ , that bisimulation implies language equivalence; for partial automata,

<sup>‡</sup>This is equivalent to the existence of a distributive law  $\lambda: TF \Rightarrow FT$  [13].

where  $F(X) = 2 \times X^A$  and  $T = 1 + -$ , that equivalence of pairs of languages, consisting of defined paths and accepted words, implies equivalence of accepted words; for probabilistic automata, where  $F(X) = [0, 1] \times X^A$  and  $T = \mathcal{D}_\omega$ , that probabilistic bisimilarity implies probabilistic/weighted language equivalence. Note that, in general, the above inclusion is strict.

## 6 Discussion

In this paper, we lifted the powerset construction on automata to the more general framework of  $FT$ -coalgebras. Our results lead to a uniform treatment of several kinds of existing and new variations of automata (that is,  $FT$ -coalgebras) by an algebraic structuring of their state space through a monad  $T$ . We showed as examples partial Mealy machines, structured Moore automata, nondeterministic, partial and probabilistic automata. Appendix A shows (as further examples) several behavioural equivalences that are extremely interesting for the theory of concurrency. It is worth mentioning that the framework instantiates to many other examples, among which *weighted automata* [27]. These are simply structured Moore automata for  $B = 1$  and  $\mathbf{T} = \mathbf{S}_\omega^-$  (for a semiring  $\mathbf{S}$ ) [8]. It is easy to see that  $\sim_{FT}$  coincides with weighted bisimilarity [5], while  $\approx_F^T$  coincides with weighted language equivalence [27].

Some of the aforementioned examples can also be coalgebraically characterized in the framework of [10]. There, instead of considering  $FT$ -coalgebras on  $\mathbf{Set}$  and  $F^*$ -coalgebras on  $\mathbf{Set}^{\mathbf{T}}$  (the Eilenberg-Moore category),  $TG$ -coalgebras on  $\mathbf{Set}$  and  $\overline{G}$ -coalgebras on  $\mathbf{Set}_{\mathbf{T}}$  (the Kleisli category) are studied. The main theorem of [10] states that under certain assumptions, the initial  $G$ -algebra is the final  $\overline{G}$ -coalgebra that characterizes (generalized) trace equivalence. In Appendix C, we present a first step in exploring the connection between both frameworks. However, the exact relationship is not clear yet and further research is needed in order to make it precise. It is worth to remark that many of our examples will not fit the framework in [10]: for instance, the exception, the side effect, the full-probability and the interactive output monads do not fulfill their requirements (the first three do not have a bottom element and the latter is not commutative). Moreover, we also note that the example of partial Mealy machines is not purely trace-like, as all the examples in [10].

There are two other future research directions. On the one hand, we will try to exploit  $F$ -bisimulations up to  $T$  [16, 17] as a sound and complete proof technique for  $\approx_F^T$ . On the other hand, we would like to lift many of those coalgebraic tools that have been developed for “branching equivalences” (such as coalgebraic modal logic [6, 26] and (axiomatization for) regular expressions [3]) to work with the “linear equivalences” induced by  $\approx_F^T$ .

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## A Beyond Bisimilarity and Traces

The operational semantics of concurrent and interactive systems is usually specified by labeled transition systems (LTS's) that are coalgebras for the functor  $\mathcal{P}_\omega(-)^A$ . A huge variety of behavioural equivalences for LTS's has been proposed in literature, since bisimilarity is often considered too strict, while trace equivalence too coarse (it is usually not compositional).

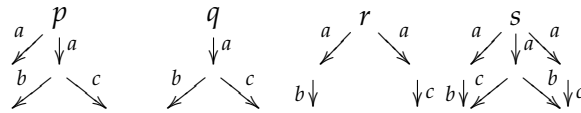
The coalgebraic bisimilarity for the functor  $\mathcal{P}_\omega(-)^A$  coincides with the ordinary notion of Park-Milner bisimilarity. Only recently, two works [10, 20] studied coalgebraic characterization for the other equivalences. The former provides a coalgebraic characterization of finite traces by employing Kleisli categories, while the latter provides a characterization of trace, failure and ready semantics by employing "behaviour objects".

In this appendix, we show that (finite) trace equivalence [11], complete trace equivalence [7], failures [4] and ready semantics [21] can be seen as instances of  $\approx_F^T$ .

Before introducing these semantics, we have to fix some notation. A labeled transition system is a pair  $\langle X, \delta \rangle$  where  $X$  is a set of states and  $\delta: X \rightarrow \mathcal{P}_\omega(X)^A$  is a function assigning to each state  $x \in X$  and to each label  $a \in A$  a set of possible successors states.  $x \xrightarrow{a} y$  means that  $y \in \delta(x)(a)$ . For a word  $w \in A^*$ ,  $x \xrightarrow{w} y$  means that  $x \xrightarrow{a_1} \dots \xrightarrow{a_n} y$  and  $w = a_1 \dots a_n$ . When  $w = \epsilon$ ,  $x \xrightarrow{\epsilon} y$  iff  $y = x$ . For a function  $\varphi \in \mathcal{P}_\omega(X)^A$ ,  $I(\varphi)$  denotes the set of all labels "enabled" by  $\varphi$ , i.e.,  $\{a \in A \text{ s.t. } \varphi(a) \neq \emptyset\}$ , while  $\text{Fail}(\varphi)$  denotes the set  $\{Z \subseteq A \text{ s.t. } Z \cap I(\varphi) = \emptyset\}$ .

Let  $\langle X, \delta \rangle$  be a LTS and  $x \in X$  be a state. A *trace* of  $x$  is a word  $w \in A^*$  such that  $x \xrightarrow{w} y$  for some  $y$ . A trace  $w$  of  $x$  is *complete* if  $x \xrightarrow{w} y$  and  $y$  stops, i.e.,  $I(\delta(y)) = \emptyset$ . A *failure pair* of  $x$  is a pair  $\langle w, Z \rangle \in A^* \times \mathcal{P}_\omega(A)$  such that  $x \xrightarrow{w} y$  and  $Z \in \text{Fail}(\delta(y))$ . A *ready pair* of  $x$  is a pair  $\langle w, Z \rangle$  such that  $x \xrightarrow{w} y$  and  $Z = I(\delta(y))$ .  $\mathcal{T}(x)$ ,  $\mathcal{CT}(x)$ ,  $\mathcal{F}(x)$  and  $\mathcal{R}(x)$  denote, respectively, the set of all traces, complete traces, failure pairs and ready pairs of  $x$ . For  $\mathcal{I}$  ranging over  $\mathcal{T}$ ,  $\mathcal{CT}$ ,  $\mathcal{F}$  and  $\mathcal{R}$ , two states  $x$  and  $y$  are  $\mathcal{I}$ -equivalent iff  $\mathcal{I}(x) = \mathcal{I}(y)$ .

For a concrete example, consider the following transition systems labeled over  $A = \{a, b, c\}$ . They are all trace equivalent because their traces are  $a, ab, ac$ . The trace  $a$  is also complete for  $p$ , but not for the others. Only  $r$  and  $s$  are failure equivalent, since  $\langle a, \{bc\} \rangle$  is a failure pair only of  $p$ , while  $\langle a, \{b\} \rangle$  and  $\langle a, \{c\} \rangle$  are failure pairs of  $p, r$  and  $s$ , but not of  $q$ . Finally they are all ready different, since  $\langle a, \emptyset \rangle$  is a ready pair only of  $p$ ,  $\langle a, \{b, c\} \rangle$  is a ready pair of  $q$  and  $s$  but not of  $r$ , and  $\langle a, \{b\} \rangle$  and  $\langle a, \{c\} \rangle$  are ready pairs only of  $r$  and  $s$ .



We are now ready to show that these equivalences are instances of  $\approx_F^T$ . We first discuss ready equivalence in details and then the others.

Take  $T = \mathcal{P}_\omega$  and  $F = \mathcal{P}_\omega(\mathcal{P}_\omega(A)) \times id^A$ . For each set  $X$ , consider the function  $\pi_X^{\mathcal{R}}: \mathcal{P}_\omega(X)^A \rightarrow FT(X)$  defined for all  $\varphi \in \mathcal{P}_\omega(X)^A$  by

$$\pi_X^{\mathcal{R}}(\varphi) = \langle \{I(\varphi)\}, \varphi \rangle \in \mathcal{P}_\omega(\mathcal{P}_\omega(A)) \times \mathcal{P}_\omega(X)^A.$$



This function allows to transform each LTS  $(X, \delta)$  into the  $FT$ -coalgebra  $(X, \pi_X^R \circ \delta)$ . The latter has the same transitions of  $\langle X, \delta \rangle$ , but each state  $x$  is “decorated” with the set  $\{I(\varphi)\}$ .

Now, by employing the powerset construction, we transform  $\langle X, \pi_X^R \circ \delta \rangle$  into the  $F$ -coalgebra  $(\mathcal{P}_\omega(X), \langle o, t \rangle)$ , where, for all  $Y \in \mathcal{P}_\omega(X)$ ,  $a \in A$ , the functions  $o: \mathcal{P}_\omega(X) \rightarrow \mathcal{P}_\omega(\mathcal{P}_\omega(A))$  and  $t: \mathcal{P}_\omega(X) \rightarrow \mathcal{P}_\omega(X)^A$  are

$$o(Y) = \bigcup_{y \in Y} \{I(\delta(y))\} \quad t(Y)(a) = \bigcup_{y \in Y} \delta(y)(a).$$

The final  $F$ -coalgebra is  $(\mathcal{P}_\omega(\mathcal{P}_\omega(A))^{A^*}, \langle \epsilon, (-)_a \rangle)$  where  $\langle \epsilon, (-)_a \rangle$  is defined as usual. The final map  $\llbracket - \rrbracket: \mathcal{P}_\omega(X) \rightarrow \mathcal{P}_\omega(\mathcal{P}_\omega(A))^{A^*}$  maps each  $\{x\}$  into a function assigning to each word  $w$ , the set  $\{Z \subseteq A \text{ s.t. } x \xrightarrow{w} y \text{ and } Z = I(\delta(y))\}$ . In other terms,  $\langle w, Z \rangle \in \mathcal{R}(x)$  iff  $Z \in \llbracket \{x\} \rrbracket(w)$ . For  $s$  being the state depicted above,  $\llbracket \{s\} \rrbracket(\epsilon) = \{\{a\}\}$ ,  $\llbracket \{s\} \rrbracket(a) = \{\{b\}, \{b, c\}, \{c\}\}$ ,  $\llbracket \{s\} \rrbracket(ab) = \llbracket \{s\} \rrbracket(ac) = \{\emptyset\}$  and for all the other words  $w$ ,  $\llbracket \{s\} \rrbracket(a) = \emptyset$ .

The coalgebraic characterization of ready semantics is summarized by the following commuting diagram:

$$\begin{array}{ccc} X & \xrightarrow{\{\cdot\}} & \mathcal{P}_\omega(X) \dashrightarrow \llbracket - \rrbracket \dashrightarrow \mathcal{P}_\omega(\mathcal{P}_\omega(A))^{A^*} \\ \pi_X^R \circ \delta \downarrow & \swarrow \langle o, t \rangle & \llbracket m \rrbracket(\epsilon) = o(m) \quad \downarrow \langle \epsilon, (-)_a \rangle \\ \mathcal{P}_\omega(\mathcal{P}_\omega(A)) \times (\mathcal{P}_\omega(X))^A & \dashrightarrow & \mathcal{P}_\omega(\mathcal{P}_\omega(A)) \times (\mathcal{P}_\omega(\mathcal{P}_\omega(\mathcal{P}_\omega(A))^{A^*}))^A \\ & & \llbracket m \rrbracket(aw) = \llbracket t(m)(a) \rrbracket(w) \end{array}$$

The other semantics can be characterized in the same way, by choosing different functors  $F$  and different functions  $\pi_X: \mathcal{P}_\omega(X)^A \rightarrow FT$ .

For failure semantics, take  $F = \mathcal{P}_\omega(\mathcal{P}_\omega(A)) \times id^A$  (as in the case of ready semantics) and  $\pi_X^F: \mathcal{P}_\omega(X)^A \rightarrow FT(X)$  defined  $\forall \varphi \in \mathcal{P}_\omega(X)^A$  as  $\langle Fail(\varphi), \varphi \rangle$ .

For both trace and complete trace equivalence, take  $F = 2 \times id^A$  (as for NDA). For trace equivalence,  $\pi_X^T: \mathcal{P}_\omega(X)^A \rightarrow FT(X)$  maps  $\varphi \in \mathcal{P}_\omega(X)^A$  into  $\langle 1, \varphi \rangle$ . Intuitively,  $(X, \delta; \pi_X^T)$  is a NDA where all the states are accepting. For complete traces,  $\pi_X^{CT}: \mathcal{P}_\omega(X)^A \rightarrow FT(X)$  maps  $\varphi$  in  $\langle 1, \varphi \rangle$  if  $I(\varphi) = \emptyset$  (and in  $\langle 0, \varphi \rangle$  otherwise).

By taking  $T = \mathcal{D}_\omega$  instead of  $T = \mathcal{P}_\omega$ , we can characterize probabilistic trace, complete trace, ready and failure as defined in [14]

## B Proof of Theorem 1

In this appendix we provide a proof of Theorem 1 by showing that there exists a functor from the category of  $FT$ -coalgebras to the category of  $F$ -coalgebras.

This functor maps each  $FT$ -coalgebra  $(X, f)$  into the  $F$ -coalgebra  $(T(X), f^\sharp)$  and each  $FT$ -homomorphism  $h: (X, f) \rightarrow (Y, g)$  into the  $F$ -homomorphism  $T(h): (T(X), f^\sharp) \rightarrow (T(Y), g^\sharp)$ . In order to prove that this is a functor we just have to show that  $T(h)$  is an  $F$ -homomorphism (i.e., the backward face of the following diagram commutes).

$$\begin{array}{ccccc}
& & T(X) & \xrightarrow{T(h)} & T(Y) \\
& \nearrow \eta_X & & & \nearrow \eta_Y \\
X & \xrightarrow{h} & Y & & \\
& \searrow f^\sharp & & & \searrow g^\sharp \\
& & FT(X) & \xrightarrow{FT(h)} & FT(Y)
\end{array}$$

(Note: In the original image, there is a vertical arrow \$f\$ from \$X\$ to \$FT(X)\$ and a vertical arrow \$g\$ from \$Y\$ to \$FT(Y)\$.)

Note that the top face of the above diagram commutes because  $h$  is an  $FT$ -homomorphism. Also the front face commutes because  $\eta$  is a natural transformation. Thus  $FT(h) \circ f^\sharp \circ \eta_X = g^\sharp \circ T(h) \circ \eta_X = h \circ g$ . Since  $\eta$  is the unit of the adjunction, then there exists a unique  $j^\sharp: T(X) \rightarrow FT(Y)$  in  $\mathbf{Set}^T$  such that  $h \circ g = j^\sharp \circ \eta_X$ . Since both  $FT(h) \circ f^\sharp$  and  $g^\sharp \circ T(h)$  are (by construction) morphisms in  $\mathbf{Set}^T$ , then  $FT(h) \circ f^\sharp = g^\sharp \circ T(h)$ .

## C Free T-Algebras

Next, we briefly restrict the study presented in Section 5 to free  $T$ -algebras. This constitutes a first step to clarify the connection of our work to the framework presented in [10].

For a monad  $T$ , the full subcategory of  $T$ -algebras consisting of the free ones is denoted by  $\mathbf{Set}_T$ , and is called the *Kleisli category* of  $T$ . The objects are sets, and a morphism  $f: X \rightarrow Y$  in  $\mathbf{Set}_T$  is a function  $f: X \rightarrow TY$  in  $\mathbf{Set}$ . The composition of two morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  in  $\mathbf{Set}_T$  is given by  $\mu_Z \circ Tg \circ f$  in  $\mathbf{Set}$ . The functor  $J: \mathbf{Set} \rightarrow \mathbf{Set}_T$  mapping a set  $X$  to itself and a function  $f: X \rightarrow Y$  to the Kleisli morphism  $Tf \circ \eta_X: X \rightarrow TY$ , has a right adjoint  $U_T$  given by  $X \mapsto TX$ . Clearly,  $U_T$  preserves limits and  $J$  preserves colimits. Moreover, if  $T: \mathbf{Set} \rightarrow \mathbf{Set}$  (weakly) preserves limits of  $\omega^{op}$ -chains, then so does  $J: \mathbf{Set} \rightarrow \mathbf{Set}_T$ .

The  $T$ -Kleisli lifting of a functor  $F: \mathbf{Set} \rightarrow \mathbf{Set}$  is a functor  $\bar{F}: \mathbf{Set}_T \rightarrow \mathbf{Set}_T$  such that  $J \circ F = \bar{F} \circ J$ . The existence of a Kleisli lifting is equivalent to the existence of a natural transformation  $\delta: FT \Rightarrow TF$  compatible with the monad structure, i.e.  $\delta \circ F\eta = \eta_F$  and  $\delta \circ F\mu = \mu_F \circ T\delta \circ \delta_T$ .

Now, let  $F$  be a functor that has a  $T$ -Kleisli lifting and that preserves the limit of the final  $\omega^{op}$ -chain, which gives the final coalgebra  $\Omega$  of  $F$ . If  $T$  is a monad which also preserves limits of  $\omega^{op}$ -chains and the final object ( $T1 \cong 1$ ), then  $J(\Omega) = \Omega$  is the final coalgebra of  $\bar{F}$ , because  $J$  preserves limits of  $\omega^{op}$ -chains. We thus have that  $\sim_F$  equivalence in  $\mathbf{Set}$  coincides with  $\sim_{\bar{F}}$  in  $\mathbf{Set}_T$ .

In order to obtain a similar result to the above for the functor  $U_T$  instead of  $J$  we need to move the position of  $T$  in our functor  $FT$  from the left to right. Let  $G$  be any endofunctor on  $\mathbf{Set}$  equipped with a distributive law  $\delta: GT \Rightarrow TG$ . If  $FT$  is naturally isomorphic to  $TG$ , and  $F\mu_X = \mu_{GX} \circ T\lambda_X$ , then  $FU_T = U_T\bar{G}$ . Since the forgetful functor  $U_T: \mathbf{Set}_T \rightarrow \mathbf{Set}$  preserves limits, if the final coalgebra  $Z$  of  $\bar{G}$  in  $\mathbf{Set}_T$  is obtained as limit of the final sequence, then we have that  $U_T(Z) = T(Z)$  is the final  $F$ -coalgebra in  $\mathbf{Set}$ .

As a concrete example of this very abstract construction take  $F = 2 \times (-)^A$  and  $G = 1 + A \times X$ . For  $T = \mathcal{P}$  we have that  $\mathcal{P}(1 + A \times -)$  is naturally isomorphic to  $2 \times \mathcal{P}(-)^A$ . It is not hard to see that all the other assumptions above are satisfied. Since the final coalgebra of

$\bar{G}$  in  $\mathbf{Set}_T$  is  $A^*$  [10], we have that  $\mathcal{P}(A^*)$  (i.e. the set of languages over  $A$ ) is the  $F$ -semantics in  $\mathbf{Set}$ , as expected.

For a monad  $T$  and an endofunctor  $G$  on  $\mathbf{Set}$ , in [10],  $TG$ -coalgebras are considered, with  $G$  having a Kleisli lifting  $\bar{G}: \mathbf{Set}_T \rightarrow \mathbf{Set}_T$ . Under the additional assumptions that  $G$ -preserves limits of  $\omega^{op}$ -chains, the monad is strong and commutative and  $\mathbf{Set}_T$  is cppo-enriched, the authors proved that the initial  $G$ -algebra  $I$  in  $\mathbf{Set}$  is a final  $\bar{G}$ -coalgebra in  $\mathbf{Set}_T$ , providing a canonical trace map  $\mathbf{tr}: X \rightarrow T(I)$  for each  $TG$ -coalgebra  $(X, g)$ . Elements  $x, y$  of  $X$  equated by the trace map are said to be trace equivalent, denoted by  $x \sim_{tr} y$ . If  $F: \mathbf{Set} \rightarrow \mathbf{Set}$  has an algebra lifting  $F^*$ , and there is a natural transformation  $\rho: TG \Rightarrow FT$ , then for every  $TG$ -coalgebra  $(X, g)$ ,  $x \sim_{tr} y$  implies  $x \approx_F^T y$  for the  $FT$ -coalgebra  $(X, \rho \circ g)$ .



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